4901. Define new coordinate variables by

$$X = \sqrt{3}x - y,$$
$$Y = x + \sqrt{3}y.$$

The equation is then straightforward.

4902. Let A have coordinates (x, y) and let the angle of projection be ϕ. Quoting the standard equation of the trajectory,

$$y = x\tan\phi - \frac{gx^2}{2u^2}(\tan^2\phi + 1).$$

For minimum launch speed, this equation, thought of as a quadratic in $\tan\phi$, must have $\Delta = 0$. Use this to show that

$$u^2 = g\left(y + \sqrt{y^2 + x^2}\right).$$

Then show that

$$\tan\phi = \frac{y}{x} + \sqrt{\left(\frac{y}{x}\right)^2 + 1}.$$

Let the angle between $\overrightarrow{OA}$ and the vertical be 2θ:

$$\cot 2\theta = \frac{y}{x}.$$

Sub in and simplify with double-angle formulae.

4903. The equations are symmetrical in $y = x$. This goes a long way to simplify things. Sketch the curves carefully, showing that there are no intersections which do not lie on $y = x$. Then solve $y = x^4 - 2x^2$ and $y = x$ simultaneously.

4904. (a) For t_0, think about the modelled reality. For k, consider the effect on the graph of increasing or decreasing k.

(b) For the model to be consistent, the total area under the graph must equal the initial size of the crowd, i.e. everyone must leave eventually. Write this as an integral statement, and carry it out. Use the substitution $\tan u = \sqrt{k}(t - t_0)$.

(c) You can express P algebraically by looking at an indefinite version of the integral in part (b). However, you don't have to. Instead, sketch by thinking about the modelled reality: consider the number of people present prior to and after $t = t_0$.

4905. The triple-angle identities, which can be proved using compound- and double-angle identities, are

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta,$$
$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta.$$

Put the LHS fractions over a common denominator. Reverse engineering a little, the best is $\cos 2\theta$. Play around with some identities until you get there. Depending on which way you go, you might need to consider $a^4 - b^4$ as a difference of two squares.

4906. Consider the boundary equation. For the LHS to be zero, at least one of the expressions must be zero. Each expression is a difference of two squares, which gives a set of six lines. The required regions form a chequerboard pattern.

4907. Since $\mathrm{f}(p) = 0$, we know that $\mathrm{f}(x)$ has a factor of $(x - p)$. Take this out, writing $\mathrm{f}(x) = (x - p)\,\mathrm{f}_1(x)$. Differentiate by the product rule to show that $\mathrm{f}_1(x)$ has a factor of $(x - p)$. Continue in this fashion.

4908. Centre the unit circle at O and the second largest circle at $(2/5, 0)$. Find the centre of the circle of radius $3/8$ by considering the distances from $(0, 0)$ and $(2/5, 0)$.

Call the centre of the third circle (p, q). Then set up three simultaneous equations in the variables p, q, r, using distances from the three centres.

4909. Sketch $y =$ LHS, by considering the behaviour just either side of each asymptote. You don't need to find SPs; in fact, there aren't any. Once you've got a good picture, the result follows easily.

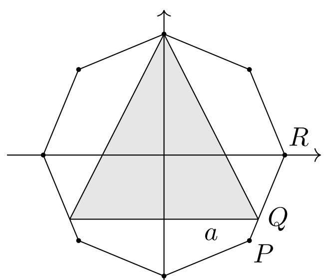
4910. (a) Set up an expression for the total distance from the ring to A and B. Substitute the equation of the ellipse in, and simplify to get 1.

(b) The velocity must be tangential to the ellipse. Differentiate implicitly to find the gradient at the point of release. Convert this to an angle, taken in the usual sense of anticlockwise from the horizontal A to B.

4911. Use the substitution $u = \ln x$, and then parts.

4912. Expand binomially and equate coefficients. Then eliminate one of the variables. You'll reach a non-analytically solvable equation. You know that you're looking for integer solutions, so you can use a sign change method. Alternatively, use N-R or fixed-point iteration.

4913. Set up:



Find the equation of line PR. Use the fact that $\tan 67.5° = 1 + \sqrt{2}$. Find the coordinates of Q in terms of a, and then the area of the triangle in terms of a. Optimise this.

4914. There is a result you can quote here. Otherwise, sketch the scenario and set up a generic circle with radius r. Find the value of r such that the circle and the curve have the same second derivative at $(0, 1)$.

4915. At the point of self-intersection,

$$\begin{aligned} \ln s \cos s &= \ln t \cos t, \\ -\ln s \sin s &= -\ln t \sin t. \end{aligned}$$

Solve these equations simultaneously. You'll need to use the fact that $s \neq t$, which yields a $... + \pi$. When you sub back in, use $\cos(\theta + \pi) \equiv -\cos\theta$. You should end up with a quadratic in s (or t).

4916. The second derivative $\text{h}''(x)$ is a quadratic. Since it has roots at $x = a$ and $x = b$, it is symmetrical around the midpoint of the two. So, the following holds for all $x \in \mathbb{R}$:

$$\text{h}''\left(\frac{a+b}{2} - x\right) = \text{h}''\left(\frac{a+b}{2} + x\right).$$

Integrate with respect to x to find a relationship involving h'. To find the constant of integration, substitute

$$x = \frac{b-a}{2}.$$

Repeat to find a relationship involving h.

4917. Find the equation of a generic tangent at $x = p$. Find the x intercept of this tangent, in terms of p. Then set up an equation in p, using the fact that tangents from a point to a circle are the same length. Solve numerically, proving (*determine* not *find*) with error bounds.

4918. The average value A is given by

$$A = \tfrac{1}{2\pi} \int_{\theta=0}^{\theta=2\pi} \cos^2\theta \sin^2\theta \, d\theta.$$

Use two double-angle formulae.

4919. (a) Actively drag out a factor of $1/n^2$, even though it isn't there. Then use a log rule.
(b) Integrate by inspection, using the result given in the question.

4920. Find the equation of a generic normal at $x = p$. Solve for re-intersections with the curve, noting that $x = p$ must be a root of the equation you set up. Find the other x value, and the y value to go with it. Use calculus to show that this y value is always at least 2.

4921. Multiply top and bottom by $\left(1 - 2^{\frac{1}{4}} + 2^{\frac{1}{2}} - 2^{\frac{3}{4}}\right)$.

4922. Consider a pair of opposite triangles, i.e. those from parallel edges of the $2n$-gon. Show that the total area of two opposite triangles is independent of the position of the central point.

4923. You can do this by brute force. But there is an easier method. Find the probability that there is at least one heart in the hand. Then calculate the expectation of the presence of at least one heart in the hand, with 1 as presence and 0 as absence. By symmetry, the same is true for all suits.

4924. Put everything onto the LHS and factorise.

4925. This is easiest combinatorially. There are 20! ways in which the numbers can be placed. First, work out the number V of vertices. Then, for successful outcomes, multiply V by the number of ways in which $\{1, 2, 3, 4, 5\}$ can be placed around a vertex, and by the number of ways in which the rest can be placed elsewhere.

— Alternative Method —

Use a conditioning approach, placing 1 without loss of generality. This does require some care: there are two cases for the placement of 2.

4926. Parts (a), (b), (c) and (e) are simple enough. Part (d) is the complicated bit.

In (d), substitute for y to produce a quadratic in x. But don't multiply it out, because it is, in fact, a quadratic in $x - d$. So, let $X = x - d$, and solve for X. This will keep the algebra under control.

Simplify as far as possible at every stage. You should be able to check you're on the right track when $X = 0$ (i.e. $x = d$) appears as a root.

4927. (a) Set up the parametric integration formula. You don't need to worry about the signing of the area here, it will come out in the wash.
(b) Integrate the terms separately, using parts. The tabular integration method is easiest for the middle term.

4928. Set up a diagram with the line of symmetry of the isosceles triangle horizontal, so that the force of magnitude B acts horizontally. The upper half is a right-angled triangle. Call its angles 2α and $90° - 2\alpha$.

Resolve along the angle bisector at both vertices, and use $\sin 2\alpha \equiv 2 \sin\alpha \cos\alpha$ to eliminate α from your equations.

4929. Factors must (in almost all cases) appear in pairs, in the form $p \times q = n$ for $p \neq q$. For example, 28 is 1×28, 2×14 and 4×7. Start by considering the cases in which n is or is not a perfect square.

4930. Find the parametric equations of the ellipse. Use these to express $A_\triangle$ in terms of a parameter t. Optimise $A_\triangle$ with respect to t.

4931. Use the substitution $q = x + y$, $p = x - y$.

4932. Use partial fractions, and write the sum longhand.

4933. Work out the angles of inclination of the other two masses, in the same sense as θ. Then consider equilibrium along the string. The tensions in the strings cancel; only the tangential components of the weight have an effect. If the three angles are measured in the same sense (as θ in the diagram), these components will appear symmetrically. Use identities to simplify and solve.

4934. The boundary equations are two circles. Sketch these, noting the point of tangency. Then consider the signs of the factors, depending on whether a point is inside or outside the circles.

4935. Quoting a standard result, the trajectory before the first bounce is

$$y = -\frac{gx^2}{2u^2} + c.$$

Work out where the first bounce is. The second trajectory is a copy of the first, translated by twice this amount. Multiply by n to find the translation after n bounces. Replace x by $(x - ...)$ in the first trajectory, and simplify.

4936. (a) Let $x = 2t$.
(b) Take out a factor of $\sec^2 x$ and then use the second Pythagorean trig identity. Simplify and then integrate by inspection.

4937. The form given is a prime factorisation. Hence, you can list the factors explicitly, and consider their sum as that of two geometric series.

4938. Solve for intersections of $xy\,(x + y) = 16$ and the boundary equation $x^2 + y^2 = 8$. Show that the curve is tangent to the circle at a single point. Hence, show that the curve is on or outside the circle.

4939. Set up the parametric integration formula. Use some identities to simplify. Most of the terms won't contribute to the value of the integral. You don't have to worry about positives and negatives in the signed area. In such a rotation they won't cancel out, giving either the total area or its negative.

4940. Let $z = x^2$. Show, using calculus to consider SPs, that the resulting quintic in z has exactly one root, which is positive.

4941. Set up the equation for intersections. For two distinct points of tangency, this equation must have two double roots. Write an identity with $(x - a)^2(x - b)^2$, and equate coefficients.

4942. (a) Consider the range of sine.
(b) Show that, as $x \to 0^+$, values keep appearing for which $\sin(\ln x) = 1$.
(c) Explain why the argument in (b) applies to any other potential limit.

4943. Draw a force diagram for the ladder containing four forces. Use circle geometry to find the angle of inclination of reaction and friction at the base. Set up three equations: horizontal, vertical and moments around the base. Find one reaction with the moments equation, and eliminate the others using horizontal and vertical equilibrium. Then manipulate the algebra using identities.

4944. Differentiate the substitution with respect to x. With some manipulation, you can separate the variables. Having got x and dx on one side and t and dt on the other, you can proceed.

4945. Prove this by construction, by finding a specific subset of S in which you can list infinitely many elements. There are all sorts of ways in which you can do this. One is as follows.

Consider the primary solution set of $\sin\theta > 1/2$, which is $(\pi/6, 5\pi/6)$. This contains the interval $[1, 2]$. So, if you can show that there are infinitely many rational squares in the set $[1, 2]$, then you have shown that S has infinitely many elements.

4946. Assume, for a contradiction, that every pair of points 1 unit apart are coloured differently. Wlog, colour a point X red. Show that all points which are $\sqrt{3}$ away from X must also be red. Use this to find a contradiction.

4947. Two are true, one is false.

4948. (a) The assumptions concern
i. friction and the mass of the string.
ii. the mass of the movable pulley.
(b) Take the masses as running $4, 2, 3$ rightwards.
Call the rightwards acceleration of the upper string over its pulley a_1 and also rightwards acceleration of the lower string over its pulley a_2. Then set up three force diagrams for the masses, writing the accelerations in terms of a_1 and a_2. You should get three equations of motion in T, a_1 and a_2.

4949. On a Venn diagram of the possibility space, there are ${}^4\text{C}_2 = 6$ regions in which two events occur, ${}^4\text{C}_1 = 4$ regions in which one event occurs, and ${}^4\text{C}_0 = 1$ region in which no events occur. So, there are 11 relevant probabilities.

(a) Consider the case in which all 11 probabilities are zero, except

$$\mathbb{P}(A \cap B) = \mathbb{P}(C \cap D) = \tfrac{k}{4},$$
$$\mathbb{P}(A' \cap B' \cap C' \cap D') = 1 - \tfrac{k}{2}.$$

(b) Set four of the two-way intersections, namely $\mathbb{P}(A \cap B)$, $\mathbb{P}(B \cap C)$, $\mathbb{P}(C \cap D)$ and $\mathbb{P}(D \cap A)$ to $\frac{k}{4}$, and set $\mathbb{P}(A' \cap B' \cap C' \cap D')$ to $1 - k$.

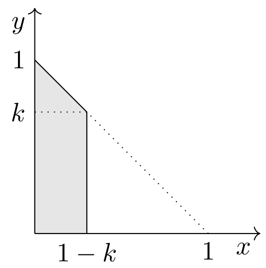
4950. Firstly, prove the half-angle formula

$$\sin^2 \tfrac{1}{2}\theta \equiv \frac{1 - \sqrt{1 - \sin^2\theta}}{2}.$$

To do this, square $\sin\theta \equiv 2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta$. Then let $s = \sin\frac{1}{2}\theta$. Use the first Pythagorean identity and the quadratic formula to solve for s^2.

Then substitute in $\frac{1}{2}\arcsin x$.

4951. At height $z = k$, the (x, y) cross-section is



Find the (x, y) area A_k of the trapezium, and then integrate A_k between $k = 0$ and $k = 1$.

4952. Find the time at which each particle sparkles, and thereby the coordinates at this point in terms of t. Consider these as a pair of parametric equations. Find the Cartesian equation.

4953. The denominator is the number of outcomes in the possibility space. So, you need to show that the numerator is the number of successful ones.

Classify the successes by the total length of the line involved, including any blank grid squares outside the counters.

4954. Define new coordinate variables:

$$p = \tfrac{1}{\sqrt{2}}(y + x),$$
$$q = \tfrac{1}{\sqrt{2}}(y - x).$$

The factors of $\frac{1}{\sqrt{2}}$ are necessary in order not to stretch the graph.

4955. (a) Draw a triangle of forces for particle A, using some circle geometry to write all of the angles in terms of θ. Use the sine rule, expanding with a compound-angle formula.

(b) Square both sides of the result from part (a) and use identities to simplify. Solve a quadratic in $\sin\theta$.

4956. (a) Deal with the cases in which $\sec x + \tan x$ is non-negative and negative separately. This will allow you to get rid of the mod sign.

(b) Let $u = \sec x$. Enact parts, and then use the second Pythagorean trig identity to split what results. You should be able to rearrange to make $2I$ the subject. Then you can use the result from part (a).

4957. Assume that $2^k + 1$ is prime, and that k can be factorised as $k = ab$ for $a, b > 1$. Find an explicit factorisation of $2^{ab} + 1$ which relies on a being odd, thus proving that a must be even.

4958. Reflection in $z = 1$ is equivalent to reflection in $z = 0$ and then translation by vector $2\mathbf{k}$. Find the equation after this. Then replace x by $x - \pi$, y by $y - 2\pi$ and z by $z + 2$. Simplify using identities.

4959. (a) i. Evaluate a definite integral with respect to a dummy variable such as p, keeping x to represent the limits of the integral $-x$ to x.

ii. It's clearer if you call the length of the ditch l, even though l will then drop out of the algebra.

(b) Write $\frac{d}{dt}(x^3) = 3x^2\frac{dx}{dt}$. Solve the resulting DE in variables x and t to show that

$$\tfrac{1}{2}x^2 + x + \ln|1 - x| = d - t.$$

In the long term, the RHS tends to $-\infty$. Work out how the RHS can possibly equal this, and make an approximation.

4960. She will see the same effect. Prove it by setting up the equation $\mathrm{f}_1''(x) - \mathrm{f}_2''(x) = 0$, which must hold for all x. Integrate it twice with respect to x.

4961. For a combinatorial approach, place A wlog. Then consider the outcomes as a list of the 5! orders of the other five letters. Success requires exactly two of D, E, F to be next to one another. For failure, all three are together or all three are separated. Count up the outcomes.

4962. The shortest distance must lie along a normal to the plane which passes through the origin. This is the line $x = y = z$.

4963. Use the fact that the golden ratio ϕ is a root of the equation $x^2 - x - 1 = 0$. This allows you to write $\phi^2 = \phi + 1$, without the need to expand surds.

4964. As suggested, the RHS is the number of ways of choosing a committee of $(r+1)$ people from a group of $(n+1)$.

Label the people $1, ..., n+1$. Then classify the ways of picking the committee by the least label picked.

4965. Wlog, let $r = 1$. Call the downwards acceleration of the upper core a. Show that each lower core starts accelerating horizontally at $\sqrt{3}a$. Resolve vertically for the upper core and horizontally for one of the lower cores. Solve simultaneously.

4966. Write e.g. $\sin x = \sin\left(2 \cdot \frac{x}{2}\right)$ and expand with a double-angle formula. The key is then to put each expression over a denominator of $\sec^2 \frac{x}{2}$.

4967. (a) The parabola that fits most closely matches the values of $\mathrm{f}(k)$, $\mathrm{f}'(k)$ and $\mathrm{f}''(k)$.

(b) Find the derivatives of the given function, and substitute into the results from part (a).

(c) Show that the approximating parabola is

$$y = -2x^2 + 5x - \ln 2 - \tfrac{3}{2}.$$

Equate to zero and solve with the quadratic formula.

4968. Rewrite the problem in coordinates $X = x+y$ and $Y = x - y$.

4969. Set up NII along the rope. Tensions are internal and cancel; only the tangential components of the weight remain. So, set up a definite integral to find the resultant force acting tangentially on the rope.

You might want to consider an approximation with a small section subtending $\delta\theta$ at the centre of the cylinder in order to set up a finite sum first. The relevant definite integral is the infinitesimal limit of this sum.

4970. The key here is: in the equation for intersections between curve and tangent line, there must be a double root at the point of tangency. If you take a squared factor out of a cubic, you must leave a linear factor behind. Express this algebraically. Multiply out and, by considering the coefficient of x^2, explain how you know that $b \in \mathbb{Z}$.

4971. When the distance from the origin is greatest or least, the tangent vector is perpendicular to the position vector. Write this algebraically, and sub it into the equation of the ellipse.

4972. Assume, for a contradiction, that

① $2^p - 1$ is prime,

② p is not prime, and can be written as $p = ab$, where a and b are integers greater than 1.

This gives $2^p - 1 = 2^{ab} - 1$. The task is to find an explicit factor of $2^{ab} - 1$.

4973. The key is the relationship between asymptotes and factors. Establish that the asymptote at $x = 0$ forces the LHS to have a factor of y. Repeat this, establishing factors from the other asymptotes. Scale the equation to put it tangent to the circle.

4974. Show that one of the centres must lie on the line of symmetry, and the the other two off it. Call the radii R, r, r. Then find the total area covered in terms of r. Show that this has only one SP, which is a minimum. Interpret this as needing analysis of the boundary cases $R = 1/2$ and $r = 1/4$. Find the area in each case.

4975. Given a particular orientation θ, work out the probability that the needle will end up crossing a crack. Then integrate this probability from $\theta = 0$ to $\theta = \pi/2$, dividing by the width of the interval.

4976. (a) Factorise fully. One double tangent passes through the origin; the other two are rotations of one another around the origin.

(b) For the double tangent through O, set up the equation for intersections. It must have two double roots apart from $x = 0$. Set up an identity and equate coefficients.

For the other tangents, it's the same, but the algebra is harder. The relevant identity is

$$\begin{aligned} &x^5 - 20x^3 + (64 - m)x - c \\ &\equiv (x+p)^2(x+q)^2(x+r). \end{aligned}$$

Equate coefficients of x^4, x^3, x^2 to get three equations in p, q, r. Then use the substitution $a = p + q$, $b = 2pq$ to solve.

4977. Unwrap the cube as a flat net, labelling the vertices (and their repetitions in the net) A, B, C, D, E, F, G, H. Map the path on the net.

4978. Use definite integration. Determine the area A_z of the triangular cross-section of the region at height z. Then integrate this with respect to z, between the appropriate limits.

4979. Factorise fully and sketch the boundary equations. Notice that no factor is repeated. This implies that crossing over any one boundary equation must change the sign of the LHS of the inequality. This will allow you, having tested e.g. the point $(1, 0)$, to shade all of the relevant regions.

4980. A combinatorial approach is easier. There are four ways to colour the central region. Consider the number of ways of colouring the outside regions, using the classification

① Type ABAC,
② Type ABAB.

4981. Consider the expression $\cos(\theta - \phi) - \cos(\theta + \phi)$.

4982. Assume, for a contradiction, limiting friction in both components. Show that this results in the skier slowing down. Then, use a triangle of forces to work out which of the components of the friction must be maximal in the limiting case. Use another triangle of forces to get the required relationship.

4983. Show first that the central hexagon can be coloured in 66 different ways. Then consider, for each of those, the colouring of the three outer triangles.

4984. Use contradiction. This is a generalisation of the proof of the irrationality of $\sqrt{2}$. The trick is to focus on individual prime factors of k.

4985. Sketch the path P of the centre of the circle. The task is to show that every point within the circle $C : x^2+y^2 = 4$ lies within a distance of 1 from path P. This is a question of analysing various cases, defining those cases by their relationship with the path P. As ever, a sketch is useful. Indeed, an accurate plot might even be warranted.

4986. The magnitude of applied couple must be equal to the magnitude of the couple formed by the weight and the reaction. Set up a force diagram without the applied couple. This is easiest if you use axes in which the prism is not rotated, but ground and forces are. Then you can formulate things using the original coordinate system. Find the equation of the normal at P, and therefore work out the moment of the weight around P.

4987. Establish that the solution set must have both of the axes as lines of symmetry. Then consider the positive quadrant. Sketch the boundary equation in that quadrant, before mirroring them elsewhere. Finally, check a point or more to work out which regions satisfy the inequality.

4988. Write down the boundary equation and factorise. The task is then a sketch. This is easier than you might think, because the magnitude of the indices means that the curve can be approximated by line segments: all points must lie very close to one of $x = 0$, $y = 0$, $x = \pm 1$, $y = \pm 1$ or $y = \pm x$.

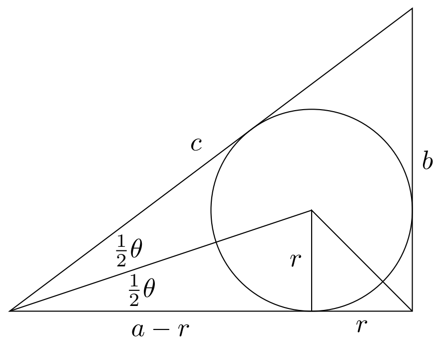
Shade the successful region, approximating it with line segments. The area can then be easily found.

4989. Consider a cubic

$$y = ax^3 + bx^2 + cx + d.$$

Completing the cube allows you to eliminate the x^2 term. This is the key step. The rest consists of keeping track of the relevant transformations. Feel free to introduce new constants as and when you need. You don't need to find explicit coefficients for the new graphs as you transform them, merely to keep track of what is zero or not, then, at the end, what is positive or negative.

4990. Use the angle in a semicircle theorem to show that $R = \frac{1}{2}c$. Then, to find r, set up the following diagram and use the given identity:



4991. (a) Show that

$$\frac{1}{u}\,du = \sec\theta\, d\theta.$$

(b) Use the substitution to write the integral as

$$\tfrac{1}{2}I = \int \sec^3\theta\, d\theta.$$

Then use integration by parts, with $u = \sec\theta$. Using the second Pythagorean trig identity, you can convert the integral so as to use the result from (a).

4992. There are various ways of doing this. I imagine it can be done algebraically. I suggest a graphical approach. If you can sketch the region defined by the inequality, then the problem is mostly done.

To sketch, factorise the boundary equation fully, noting its rotational symmetry around the origin. You might use variables $\frac{1}{\sqrt{2}}(x+y)$ and $\frac{1}{\sqrt{2}}(x-y)$. Having sketched, show that there is exactly one circle centred at the origin which remains fully within the region defined by the inequality. Find its radius by considering the self-intersections of the boundary equation.

— Nota Bene —

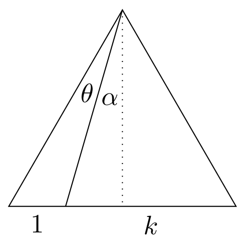
This problem offers almost no way of visualising the solution before you have the solution. In that respect it's a tough one. But that's the power of visual approaches to algebraic problems. Like the ancient Greeks, I err on the side of geometry!

4993. Consider the set

$$S = \{ax + by : x, y \in \mathbb{Z} \text{ and } ax + by > 0\}.$$

S is a set of positive integers. So, it must have a smallest element $s = ap + bq$. You need to show that $s = 1$. Do this by showing that s is a divisor of both a and, by the same argument, b.

4994. Consider the pair of spheres as a single object. It has three forces acting on it: two reactions acting radially, and the combined weight. Both reactions pass through the centre of the bowl. Hence, so must the weight, meaning that the centre of mass of the combined object must lie directly below O, on the intersection of the vertical and PQ. This reduces the problem to a geometric one:



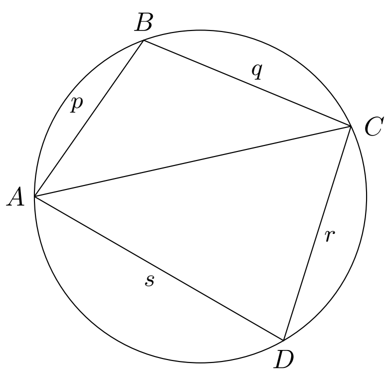
Find $\tan\alpha$ in terms of k, then use a compound-angle formula to expand $\tan\theta = \tan(30° - \alpha)$.

4995. By sketching carefully and considering symmetry, find the equation $y = mx$ of the line which passes through the origin and the point of intersection. Calculate the area by definite integration, and then use half/double-angle formulae.

4996. This question is about visualising the possibility space. With the right visualisation, the problem requires very little in the way of working. Start with $n = 1, 2, 3, ...$, all of which have possibility space which can be visualised in 3D space. Then extend the argument into higher dimensions.

The final result is simple and, in some senses, could be written down with a "clearly". The difficulty is finding a way to express the argument rigorously. There are lots of ways to do this, of which iterated integration is one.

4997. Split the quadrilateral into two triangles:



Calculate the areas of these two using $\frac{1}{2}ab\sin C$, then simplify using trig identities. Write the whole thing in terms of side lengths and the cosine of a single interior angle.

Then, use the cosine rule on both triangles and eliminate the length of the chord. This will allow you to express the cosine in your earlier formula in terms of side lengths. From this point, you have the area in terms of the side lengths, so the rest is an exercise in algebraic manipulation.

4998. As with many problems involving Cartesian axes or functions, the first task is to sketch the scenario carefully, so as to get a clear mental image of the relevant mathematics. In this case, such sketching requires a change of coordinate system. Express the problem in terms of the variables

$$X = x + y,$$
$$Y = x - y.$$

These run along axes aligned at 45° to x and y.

Once you've done this, you're looking to prove that the area in (X, Y) space tends to 4 (there's a scale factor associated with the change of variables). To do this, divide and conquer:

① The total area in the first, second and fourth quadrants can be analysed using integration. Show that it tends to 3.

② The area in the third quadrant $x, y < 0$ can be analysed by comparing it to the area of a kite. Show that it tends to 1.

4999. There are a number of ways in which this can be proved, including brute force rearrangement. My preferred method, which I think comes closest to showing the *why* of the question, is to prove that changing the position of (a, b) doesn't change the total shaded area. Combined with the fact that the result is trivially true when (a, b) is the centre of the circle, this gives the result.

So, place (a, b) at an arbitrary point, and show that translating (a, b) in the x direction maintains area. For visualisation, consider a small change in position to $(a + \delta x, b)$. You then need to work in the limit as $\delta x \to 0$. In the limit, area is generated and destroyed at the straight-edge boundaries of the shaded regions. The task is to count up the contributions.

Show that the total contribution of the vertical chord is proportional to b. You can then generalise this argument to the other chords, representing all of the contributions as lengths. Once you've got the relevant lengths, it is elementary geometry to show that they sum to zero.

5000. You need to prove that the shortest path between each pair of curves is the same length.

This is best approached by careful sketching, so that you have a very clear picture of where you're trying to get. Particularly, find out everything you can behaviour-wise about the quartic. Then, work out ① the distance between the circle and the parabola, using a generic normal to the curve at $x = a$, and ② the distance between the parabola and the quartic.

Then, ③ to *find* the shortest path from the circle to the quartic isn't too difficult, by guessing from a sketch or using numerical methods. The last part, and the hardest, is *proving* that the distance you have found is the shortest one.

You might want to consider a "middleman", i.e. an analysable curve which you can use to compare the circle and the quartic.

——— End of Volume V ———